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Function spaces and a property of Reznichenko ☆

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Abstract

In this paper we show that for a set X of real numbers the function space $C_p(X)$ has both a property introduced by Sakai in [Proc. Amer. Math. Soc. 104 (1988) 917–919] and a property introduced by Reznichenko (see [Topology Appl. 104 (2000) 181–190]) if and only if all finite powers of X have a property that was introduced by Gerlits and Nagy in [Topology Appl. 14 (1982) 151–161]. It follows that the minimal cardinality of a set of real numbers for which the function space does not have the properties of Sakai and Reznichenko is equal to the additivity of the ideal of first category sets of real numbers. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

We consider spaces which are at least $T_{3\frac{1}{2}}$. The reason for this separation hypothesis is that it implies the existence of many continuous real-valued functions. We consider the set $C(X)$ of continuous real-valued functions on X as a subset of the set of all real-valued functions on X. The latter set carries a natural topology, namely the Tychonoff product topology when considering it as the product of X copies of the real line. $C(X)$ inherits this topology, which is also called the topology of pointwise convergence. The symbol $C_p(X)$ denotes $C(X)$, endowed with the topology of pointwise convergence. The symbol

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$\underline{o}$ denotes the constantly zero function. Since $C_p(X)$ is a topological vector space we may confine ourselves to the point $\underline{o}$ when studying local properties.

For X a space and for $x \in X$ the symbol Ω_x denotes the set $\{A \subset X \setminus \{x\}\colon x \in \overline{A}\}$. According to [7] the following property for a point $x \in X$ was introduced by Reznichenko:

For each $A \in \Omega_x$ there is a countably infinite pairwise disjoint collection $\mathcal{F}$ of finite subsets of A such that for each neighborhood U of x, for all but finitely many $F \in \mathcal{F}$, $U \cap F$ is nonempty.

In [7] this property is called the weak Fréchet–Urysohn property at x; we shall say that X has the *Reznichenko property at* x. When X has the Reznichenko property at each of its points, then we say that X has the *Reznichenko property*.

Some of the results in this paper and in some of our references can be stated conveniently in terms of the following selection principle: Let S be an infinite set and let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of S. Then $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle:

For each sequence $(A_n\colon n \in \mathbb{N})$ of elements of $\mathcal{A}$, there is a sequence $(B_n\colon n \in \mathbb{N})$ such that for each n $B_n \in A_n$ and $\{B_n\colon n \in \mathbb{N}\} \in \mathcal{B}$.

In [10] Sakai introduced the following notion: X has *countable strong fan tightness at* x if the selection principle $S_1(\Omega_x, \Omega_x)$ holds. If X has countable strong fan tightness at each of its elements, then we say that X has *countable strong fan tightness*. According to [2] an open cover $\mathcal{U}$ of X is said to be an *ω-cover* if $X \notin \mathcal{U}$ and if for each finite set $F \subset X$ there is a $U \in \mathcal{U}$ such that $F \subset U$. The collection of all ω-covers of X will be denoted by the symbol Ω. The collection of all open covers of X will be denoted by $\mathcal{O}$. The property $S_1(\mathcal{O}, \mathcal{O})$ was introduced by Rothberger in 1938 in [9]. Sakai proved in [10] that:

Theorem 0.1 (Sakai). *For a $T_{3\frac{1}{2}}$-space X, the following are equivalent*:

(1) *$C_p(X)$ has property $S_1(\Omega_{\underline{o}}, \Omega_{\underline{o}})$.*
(2) *X has property $S_1(\Omega, \Omega)$.*
(3) *For each $n \in \mathbb{N}$, X^n has property $S_1(\mathcal{O}, \mathcal{O})$.*

In [3] Hurewicz introduced the following property, now called the *Hurewicz property*: For each sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n\colon n \in \mathbb{N})$ of finite sets such that for each n $\mathcal{V}_n \subset \mathcal{U}_n$, and for each $x \in X$, for all but finitely many n, $x \in \bigcup \mathcal{V}_n$.

In [2] the authors introduced a property denoted $(*)$. X is said to have *property* $(*)$ if there is for each sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of open covers of X a decomposition $X = \bigcup_{n \in \mathbb{N}} X_n$ such that:

For each n and m there are $k < \ell$ such that $m < k$ and there are $U_i \in \mathcal{U}_i$, $k \leqslant i \leqslant \ell$, such that $X_n \subset \bigcup_{k \leqslant i \leqslant \ell} U_i$.

In [8] it was given an alternate characterization of the property $(*)$ which is more convenient for our purposes:

Theorem 0.2 (Nowik–Scheepers–Weiss). *For a space X the following are equivalent*:

(1) *X has property $(*)$.*

(2) *For each sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of open covers of X there is a sequence $(U_n\colon n \in \mathbb{N})$ and there is an increasing function $f:\mathbb{N} \to \mathbb{N}$ such that for each n the set U_n belongs to $\mathcal{U}_n$ and for each $x \in X$, for all but finitely many n, $x \in \bigcup_{f(n)\leqslant i<f(n+1)} U_i$.*

(3) *X has both the Hurewicz property and property $S_1(\mathcal{O},\mathcal{O})$.*

The property in (2) of Theorem 0.2 is also called *$add(\mathcal{M})$-small* in [8]; the reason for this is:

Theorem 0.3 (Nowik–Scheepers–Weiss). *The minimal cardinality of a set of real numbers not having property $(*)$ is equal to the minimal cardinality of a family of first category sets whose union is not first category.*

This latter cardinal number is called the *additivity of the ideal of first category sets*, and is denoted by the symbol $add(\mathcal{M})$.

Our main result is:

Main Theorem. *For a $T_{3\frac{1}{2}}$-space X the following are equivalent*:

(1) *Each finite power of X has property $(*)$.*

(2) *$C_p(X)$ has countable strong fan tightness as well as Reznichenko's property.*

The paper is organized as follows: In the first section we characterize having $(*)$ in all finite powers in terms of properties of ω-covers of X. In the second section we characterize the second property in Main Theorem in terms of a more convenient selection principle. In Section 3 we derive the main theorem and some corollaries.

1. Property $(*)$ in all finite powers

An open cover $\mathcal{U}$ of X is said to be a *large cover* if it is countably infinite, and for each $x \in X$ the set $\{U \in \mathcal{U}\colon x \in U\}$ is infinite. In [13] a space X was said to have the *grouping property* if for each large cover $\mathcal{U}$ of X there is a sequence $(\mathcal{V}_n\colon n \in \mathbb{N})$ of finite pairwise disjoint sets such that for each n $\mathcal{V}_n \subset \mathcal{U}$, and for each $x \in X$, for all but finitely many n, $x \in \bigcup \mathcal{V}_n$. It was shown in Lemma 3 of [13] that if X has the Hurewicz property, then it has the grouping property.

We shall say that X has the *ω-grouping property* if: for each ω-cover $\mathcal{U}$ of X there is a sequence $(\mathcal{V}_n\colon n \in \mathbb{N})$ of pairwise disjoint finite subsets of $\mathcal{U}$ such that for each finite set $F \subset X$, for all but finitely many n, there is a $V \in \mathcal{V}_n$ such that $F \subset V$.

Lemma 1.1. *If each finite power of X has the Hurewicz property, then X has the ω-grouping property.*

Proof. Let $\mathcal{U}$ be an ω-cover of X. Then $\mathcal{U}^* := \{U^n: n \in \mathbb{N}, U \in \mathcal{U}\}$ is a large cover of $\sum_{n\in\mathbb{N}} X^n$. Since each X^n has the Hurewicz property, also $\sum_{n\in\mathbb{N}} X^n$ has the Hurewicz property. By [13, Lemma 3], $\sum_{n\in\mathbb{N}} X^n$ has the grouping property. Thus, let $(\mathcal{V}_m: m \in \mathbb{N})$ be a sequence of pairwise disjoint finite subsets of $\mathcal{U}^*$ such that for each $x \in \sum_{n\in\mathbb{N}} X^n$, for all but finitely many n there is a $V \in \mathcal{V}_n$ with $x \in V$.

For each n, put $\mathcal{E}_n := \{U \in \mathcal{U}$: for some m, $U^m \in \mathcal{V}_n\}$. Then $(\mathcal{E}_n: n \in \mathbb{N})$ witnesses for $\mathcal{U}$ that X has the ω-grouping property. □

The space X is said to have the $\omega - (*)$ *property* if for each sequence $(\mathcal{U}_n: n \in \mathbb{N})$ of ω-covers of X there are a sequence $(U_n: n \in \mathbb{N})$ and an increasing function $f: \mathbb{N} \to \mathbb{N}$ such that:

(i) For each n, $U_n \in \mathcal{U}_n$.
(ii) If $m \neq n$, then $U_m \neq U_n$.
(iii) For each finite $F \subset X$, for all but finitely many n there is an $i \in [f(n), f(n+1))$ with $F \subset U_i$.

These two properties will now be used to give alternate descriptions of the property of having $(*)$ in all finite powers. The following two lemmas will play a role in our characterization.

Lemma 1.2. *For each $n \in \mathbb{N}$ let X_n be a subspace of the space X such that each X_n has property $(*)$. Then $\bigcup_{n\in\mathbb{N}} X_n$ has property $(*)$.*

Proof. Let $(\mathcal{U}_n: n \in \mathbb{N})$ be a sequence of open covers of $Y = \bigcup_{n\in\mathbb{N}} X_n$. Write $\mathbb{N} = \bigcup_{n\in\mathbb{N}} Z_n$, where each Z_n is infinite and for $m \neq n$, $Z_m \cap Z_n = \emptyset$. For a fixed n, if $m \in Z_n$, then let m^+ denote the first element of Z_n which is larger than m.

For each n consider X_n and the sequence $(\mathcal{U}_m: m \in Z_n)$ of open covers of X_n. By Theorem 0.2 there exist a sequence $(U_m: m \in Z_n)$ and an increasing function $f_n: Z_n \to Z_n$ such that for each m we have $U_m \in \mathcal{U}_m$, and for each $x \in Z_n$, for all but finitely many $m \in Z_n$ there is a $j \in Z_n$ with $f_n(m) \leqslant j < f_n(m^+)$ and $x \in U_j$.

Now choose an increasing function $f: \mathbb{N} \to \mathbb{N}$ such that for each n we have: For all but finitely many k there is an $m \in Z_n$ such that $f(k) < f_n(m) < f_n(m^+) < f(k+1)$. Then the function f and the sequence $(U_n: n \in \mathbb{N})$ witness for $(\mathcal{U}_n: n \in \mathbb{N})$ that Y has property $(*)$. □

The next lemma shows that in the definition of property $(*)$ we may restrict ourselves to the case where all $\mathcal{U}_n$ are ω-covers.

Lemma 1.3. *For a space X the following are equivalent*:

(1) *X has property $(*)$;*
(2) *For each sequence $(\mathcal{U}_n: n \in \mathbb{N})$ of ω-covers of X there are an increasing function $f: \mathbb{N} \to \mathbb{N}$ and a sequence $(U_n: n \in \mathbb{N})$ such that for each n $U_n \in \mathcal{U}_n$, and for each $x \in X$, for all but finitely many m there is a $j \in [f(m), f(m+1))$ such that $x \in U_j$.*

Proof. We must show that (2) $\Rightarrow$ (1). Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of open covers of X and let $(Y_n\colon n \in \mathbb{N})$ be a sequence of infinite pairwise disjoint subsets of $\mathbb{N}$ such that $\mathbb{N} = \bigcup_{n\in\mathbb{N}} Y_n$.

Fix an n and let $k_1 < k_2 < \cdots < k_m < \cdots$ be a bijective enumeration of Y_n. Define $\mathcal{V}_n$ to be the set

$$\{U_1 \cup \cdots \cup U_m\colon m \in \mathbb{N} \text{ and } U_i \in \mathcal{U}_{k_i},\ i \leqslant m\}.$$

Then $(\mathcal{V}_n\colon n \in \mathbb{N})$ is a sequence of ω-covers of X. Applying (2) to this sequence, we find an increasing function $f:\mathbb{N} \to \mathbb{N}$ and a sequence $(V_j\colon j \in \mathbb{N})$ such that for each j, $V_j \in \mathcal{V}_j$, and for each x, for all but finitely many n, $x \in \bigcup_{f(n)\leqslant j<f(n+1)} V_j$.

Each V_j is of the form $U_{k_1^j} \cup \cdots \cup U_{k_{m_j}^j}$, where $k_1^j < \cdots < k_{m_j}^j$ are elements of Y_j. For each n put $b_n = \min\{k_1^j\colon f(n) \leqslant j < f(n+1)\}$ and put $t_n = \max\{k_{m_j}^j\colon f(n) \leqslant j < f(n+1)\}$. Then choose an increasing function $g:\mathbb{N} \to \mathbb{N}$ such that for all but finitely many n there is an m with $g(n) < b_m < t_m < g(n+1)$. For s equal to some $k_{m_j}^j$, put $U_s = U_{k_{m_j}^j}$; for other values of s, let $U_s \in \mathcal{U}_s$ be arbitrary.

Then, by Theorem 0.2, g and the sequence $(U_s\colon s \in \mathbb{N})$ witness for $(\mathcal{U}_s\colon s \in \mathbb{N})$ that X has property $(*)$. $\square$

Theorem 1.4. *For a space X the following are equivalent*:

(1) *X has the $\omega - (*)$ property*;

(2) *All finite powers of X have property $(*)$*;

(3) *X has property $S_1(\Omega, \Omega)$ as well as the ω-grouping property.*

Proof. (1) $\Rightarrow$ (2): We use Lemma 1.3. Fix an $n \in \mathbb{N}$. Let $(\mathcal{U}_m\colon m \in \mathbb{N})$ be a sequence of ω-covers of X^n. For each m let $\mathcal{V}_m$ be an ω-cover of X such that $\{V^n\colon V \in \mathcal{V}_m\}$ is a refinement of $\mathcal{U}_m$ (see [4, Lemma 3.3]). By (1) choose a sequence $(V_n\colon n \in \mathbb{N})$ and an increasing function $f:\mathbb{N} \to \mathbb{N}$ such that for each m, $V_m \in \mathcal{V}_m$, and for each finite $F \subset X$, for all but finitely many n, there is a $j \in [f(n), f(n+1))$ with $F \subset V_j$. Then for each j choose a $U_j \in \mathcal{U}_j$ such that $V_j^n \subseteq U_j$. The function f and the sequence $(U_n\colon n \in \mathbb{N})$ witness for $(\mathcal{U}_n\colon n \in \mathbb{N})$ that X^n has property $(*)$.

(2) $\Rightarrow$ (3): Since property $(*)$ implies $S_1(\mathcal{O}, \mathcal{O})$, (2) implies that each finite power of X has property $S_1(\mathcal{O}, \mathcal{O})$, and so by Sakai's theorem, X has property $S_1(\Omega, \Omega)$. By Theorem 0.2, $(*)$ implies the Hurewicz property. Thus, by (2), all finite powers of X have the Hurewicz property. Lemma 1.1 implies that X has the ω-grouping property.

(3) $\Rightarrow$ (1): Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of ω-covers of X. Since X has property $S_1(\Omega, \Omega)$, we may choose for each n a $U_n \in \mathcal{U}_n$ such that $\mathcal{U} = \{U_n\colon n \in \mathbb{N}\}$ is an ω-cover of X. Now apply the ω-grouping property to the ω-cover $\mathcal{U}$. We find for each m a finite set $\mathcal{V}_m \subset \mathcal{U}$ such that for $m \neq n$, $\mathcal{V}_m \cap \mathcal{V}_n = \emptyset$, and for each finite set $F \subset X$, for all but finitely many n there is a $V \in \mathcal{V}_n$ such that $F \subset V$.

Now choose an increasing function $f:\mathbb{N} \to \mathbb{N}$ such that for each n there is an m with the property that if $U_k \in \mathcal{V}_m$, then $f(n) \leqslant k < f(n+1)$. Then f and the sequence $(U_n\colon n \in \mathbb{N})$ witness for $(\mathcal{U}_n\colon n \in \mathbb{N})$ that X has the $\omega - (*)$ property. $\square$

2. $C_p(X)$, $S_1(\Omega_{\underline{o}}, \Omega_{\underline{o}})$ and Reznichenko's property

The following strengthening of $S_1(\Omega_x, \Omega_x)$ will be important in this section:

For each sequence $(A_n\colon n \in \mathbb{N})$ of elements of Ω_x there are an increasing function $f : \mathbb{N} \to \mathbb{N}$ and a sequence $(b_n\colon n \in \mathbb{N})$ such that for each n we have $b_n \in A_n$ and for each neighborhood U of x, for all but finitely many n, $U \cap \{b_j\colon f(n) \leqslant j < f(n+1)\} \neq \emptyset$.

This property will be denoted $S_1^*(\Omega_x, \Omega_x)$. It is evident that this property implies both the Reznichenko property (let $A \in \Omega_x$ be given; for each n set $A_n = A$) and countable strong fan tightness. The converse of this implication is also true:

Theorem 2.1. *For an infinite T_1-space X and for $x \in X$ the following are equivalent*:
(1) *X has property $S_1^*(\Omega_x, \Omega_x)$.*
(2) *X has both the Reznichenko property at x and property $S_1(\Omega_x, \Omega_x)$.*

Proof. We must show that (2) $\Rightarrow$ (1). Let $(A_n\colon n \in \mathbb{N})$ be a sequence of elements of Ω_x. First apply $S_1(\Omega_x, \Omega_x)$ to obtain a sequence $(b_n\colon n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $B := \{b_n\colon n \in \mathbb{N}\} \in \Omega_x$. Apply the Reznichenko property to B to find a sequence $(V_n\colon n \in \mathbb{N})$ of pairwise disjoint finite subsets of B such that for each neighborhood U of x, for all but finitely many n, $V_n \cap U \neq \emptyset$. Then choose an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that for each n there is an m with the property that for each $a \in V_m$ the minimal k with $a = b_k$ is in $[f(n), f(n+1))$. Then f and the sequence $(b_n\colon n \in \mathbb{N})$ witness for $(A_n\colon n \in \mathbb{N})$ that X has property $S_1^*(\Omega_x, \Omega_x)$. □

3. The main result

We now prove Main Theorem, stated in the introduction.

(1) $\Rightarrow$ (2): Assume that each finite power of X has property $(*)$. Then by Theorem 1.4 X has property $S_1(\Omega, \Omega)$ as well as the ω-grouping property. By Sakai's theorem $C_p(X)$ has property $S_1(\Omega_{\underline{o}}, \Omega_{\underline{o}})$. Thus, by Theorem 2.1 we only need to show that $C_p(X)$ has Reznichenko's property. Thus, let $A \in \Omega_{\underline{o}}$ be given. We may assume that A is countable, since the tightness of $C_p(X)$ is countable. Enumerate A bijectively as $(f_n\colon n \in \mathbb{N})$. Let $(g_n\colon n \in \mathbb{N})$ be a sequence of nowhere negative continuous functions which converges pointwise to $\underline{o}$ such that:
(a) If $m \neq n$, then for all x, $g_m(x) \cdot g_n(x) = 0$.
(b) For each m there is an $x \in X$ such that $g_m(x) = 1$.

For each n put $h_n = |f_n| + g_n$. Then $\{h_n\colon n \in \mathbb{N}\} \in \Omega_{\underline{o}}$.

On X define for each n and m:

$$U_n^m := \left\{x \in X\colon h_n(x) < \frac{1}{m}\right\}.$$

Then for each m the set $\mathcal{U}_n := \{U_n^m\colon n \in \mathbb{N}\}$ is an ω-cover of X. By Theorem 1.4 X has property $S_1(\Omega, \Omega)$. By [12, Theorem 2] player ONE of the following game does not have a winning strategy: ONE and TWO play an inning per $n \in \mathbb{N}$: In the nth inning ONE first chooses an ω-cover O_n of X, and TWO then responds with a $T_n \in O_n$. A play

$$O_1, T_1, \ldots, O_n, T_n, \ldots$$

is won by TWO if $\{T_n\colon n \in \mathbb{N}\}$ is an ω-cover of X; otherwise, ONE wins.

For the given sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of ω-covers of X, define the following strategy for ONE: In the first inning ONE plays $O_1 := \mathcal{U}_1$. TWO responds with a $U^1_{m_1} \in O_1$. Then, in the second inning, ONE plays $O_2 := \{U^2_m\colon m > m_1\} \setminus \{U^1_{m_1}\}$; TWO responds with a $U^2_{m_2} \in O_2$. In the third inning ONE plays $O_3 := \{U^3_m\colon m > m_2\} \setminus \{U^1_{m_1}, U^2_{m_2}\}$, and TWO responds with some $U^3_{m_3} \in O_3$, and so on.

Since this is not a winning strategy for ONE, we obtain a sequence

$$U^1_{m_1}, U^2_{m_2}, \ldots, U^k_{m_k}, \ldots$$

such that:

(i) $i < j \Rightarrow m_i < m_j$ and $U^i_{m_i} \neq U^j_{m_j}$;

(ii) $\mathcal{U} := \{U^i_{m_i}\colon i \in \mathbb{N}\}$ is an ω-cover of X.

Since by Theorem 1.4 X has the ω-grouping property, find for each m a finite subset $\mathcal{V}_m$ of $\mathcal{U}$ such that these finite sets are pairwise disjoint and for each finite set $F \subset X$, for all but finitely many m, there is a $V \in \mathcal{V}_m$ such that $F \subset V$.

For each k define

$$G_k := \{f_{m_i}\colon U^i_{m_i} \in \mathcal{V}_k\}.$$

Then the sequence $(G_k\colon k \in \mathbb{N})$ is a sequence of finite, pairwise disjoint subsets of A. Let W be a neighborhood of $\underline{o}$. We may assume that W is a basic neighborhood, and thus for some $\varepsilon > 0$ and for some finite subset F of X,

$$W = \{f \in C_p(X)\colon x \in F \Rightarrow |f(x)| < \varepsilon\}.$$

Choose $n_1 \in \mathbb{N}$ large enough so that for each $j \geqslant n_1$ there is a $U^{i_j}_{m_{i_j}} \in \mathcal{V}_j$ with $F \subset U^{i_j}_{m_{i_j}}$. Then choose $n_2 > n_1$ so large that for all $j \geqslant n_2$ we have $1/m_{i_j} < \varepsilon$. Then for all $j \geqslant n_2$ we have $G_j \cap W \neq \emptyset$.

Thus the sequence $(G_k\colon k \in \mathbb{N})$ witnesses for A that $C_p(X)$ has the Reznichenko property. This completes the proof of (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1): Assume that $C_p(X)$ has the Reznichenko property as well as property $S_1(\Omega_{\underline{o}}, \Omega_{\underline{o}})$. By Sakai's theorem X has property $S_1(\Omega, \Omega)$. Thus, by Theorem 1.4 we only need to show that X has the ω-grouping property. Let $\mathcal{U}$ be an ω-cover of X.

Since X has property $S_1(\Omega, \Omega)$ we may assume that $\mathcal{U}$ is countable, and by [11, Lemma 21] we may choose for each n an ω-cover $\mathcal{U}_n \subset \mathcal{U}$ such that for $m \neq n$ we have $\mathcal{U}_m \cap \mathcal{U}_n = \emptyset$.

For each m define an $A_m \in \Omega_{\underline{o}}$ as follows:

For $F \subset X$ finite and for $U \in \mathfrak{U}_m$ with $F \subset U$, choose a continuous function $\Phi(F, U)$ from X to [0,1] such that:

(1) $\Phi(F, U)[F] = \{0\}$, and

(2) $\Phi(F, U)[X \setminus F] = \{1\}$.

$A_m := \{\Phi(F, U)\colon F \subset X$ finite, $U \in \mathcal{U}_m$ and $F \subset U\}$.

By Theorem 2.1 choose an increasing function $f : \mathbb{N} \to \mathbb{N}$ and choose for each n a $\Phi(F_n, U_n) \in A_n$ such that for each neighborhood W of $\underline{o}$, for all but finitely many m there is an $n \in [g(m), g(m+1))$ with $\Phi(F_n, U_n) \in W$.

Observe that if $\Phi(F_n, U_n) \in A_n$, then $U_n \in \mathcal{U}_n$, and as the $\mathcal{U}_n$s are pairwise disjoint, the U_ns are pairwise distinct. Thus, if for each m we set

$$\mathcal{V}_m = \bigl\{U_n\colon g(m) \leqslant n < g(m+1)\bigr\},$$

then the finite sets $\mathcal{V}_m \subset \mathcal{U}$ are pairwise disjoint. Moreover, these witness for $\mathcal{U}$ that X has the ω-grouping property. This completes the proof of (2) $\Rightarrow$ (1).

During the proof of Corollary 3.1 we shall need a result of A.W. Miller regarding the cardinal number $add(\mathcal{M})$. Let $cov(\mathcal{M})$ denote the minimal cardinality of a family of first category sets whose union covers the real line. For functions f and g from $\mathbb{N}$ to $\mathbb{N}$ define: $f \prec g$ if $\lim_{n\to\infty}(g(n) - f(n)) = \infty$. Then $\prec$ is a partial ordering of this set of functions. A set $\mathcal{F}$ of such functions is *bounded* if there is a function g such that $f \prec g$ whenever $f \in \mathcal{F}$; otherwise, it $\mathcal{F}$ is said to be *unbounded*. The symbol $\mathfrak{b}$ denotes the minimal cardinality of an unbounded set of such functions. According to a well-known theorem of A.W. Miller,

$$add(\mathcal{M}) = \min\bigl\{\mathfrak{b}, cov(\mathcal{M})\bigr\}.$$

Corollary 3.1. *Let κ be an infinite cardinal number. Then the following are equivalent*:

(1) $\kappa < add(\mathcal{M})$.

(2) *For each space X of countable tightness and for each $x \in X$, if $\chi(x, X) \leqslant \kappa$, then X has both properties $S_1(\Omega_x, \Omega_x)$ and the Reznichenko property at x.*

(3) *For each set X of real numbers of cardinality κ, $C_p(X)$ has both Reznichenko's property and property $S_1(\Omega_{\underline{o}}, \Omega_{\underline{o}})$.*

Proof. (1) $\Rightarrow$ (2): Let $x \in X$ be such that $\chi(x, X) = \kappa$. Since $add(\mathcal{M}) \leqslant cov(\mathcal{M})$, we have $\kappa < cov(\mathcal{M})$, the minimal number of first category sets needed to cover the real line. By [12, Theorem 13B], X has property $S_1(\Omega_x, \Omega_x)$ at x.

Towards verifying that X has the Reznichenko property at x, let $(U_\alpha\colon \alpha < \kappa)$ enumerate a neighborhood base of x. Consider an $A \in \Omega_x$. Since X has countable tightness, we may assume that A is countable. Enumerate A bijectively as $(a_n\colon n \in \mathbb{N})$. For each $\alpha < \kappa$ and for each $n \in \mathbb{N}$ define

$$f_\alpha(n) = \min\{m > n\colon a_m \in U_\alpha\}.$$

Then $\{f_\alpha\colon \alpha < \kappa\}$ is a set of functions from $\mathbb{N}$ to $\mathbb{N}$. Since κ is smaller than $add(\mathcal{M})$, it is smaller than $\mathfrak{b}$. Thus this family of functions is bounded. Let g be an increasing function such that $g(0) > 1$ and for each α, $\lim_{n\to\infty}(g(n) - f_\alpha(n)) = \infty$. Then define a function h so that $h(0) = g(0)$ and for each n, $h(n+1) = g(h(n))$.

Since g is increasing, so is h. Thus, for each α, for all but finitely many n, $f_\alpha(h(n)) < g(h(n)) = h(n+1)$. For each n put $F_n = \{a_j\colon h(n) \leqslant j < h(n+1)\}$. Then the sequence $(F_n\colon n < \infty)$ witnesses the Reznichenko property for A.

(2) $\Rightarrow$ (3): Since X is a set of real numbers, the Arhangel'skiĭ–Pytkeev theorem (see [1, Theorem II.1.1]) implies that $C_p(X)$ has countable tightness. It is evident that for each $f \in C_p(X)$ we have $\chi(C_p(X), f) = |X| = \kappa$. Now apply (2).

(3) $\Rightarrow$ (1): Let X be a set of real numbers of cardinality κ. By (3), $C_p(X)$ has both Reznichenko's property and property $S_1(\Omega_{\underline{o}}, \Omega_{\underline{o}})$. By Main Theorem each finite power of X has property $(*)$. Thus, each set of real numbers of cardinality κ has property $(*)$. By Theorem 0.3, $\kappa < add(\mathcal{M})$. $\square$

Conjecture. $\mathfrak{b}$ is the minimal cardinality of a set X of real numbers such that $C_p(X)$ does not have the Reznichenko property.

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